Diffraction corrections in radiometry: spectral and total power and asymptotic properties

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Wolf's result for integrated flux in the case of diffraction by a circular lens or aperture in the scalar, paraxial Fresnel approximation is considered anew. Compact integral formulas for pertinent infinite sums are derived, and the result's generalizations to extended sources and Planckian sources and asymptotic aspects at small wavelength and high temperature are all considered. Simplification of calculations for an actual absolute radiometer is demonstrated.

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1. INTRODUCTION

In radiometry, one often studies an optical setup with a coaxial circular source, circular aperture or lens, and circular detector, such as that shown in Fig. 1. One may desire to learn the source radiance L, source spectral radiance $L_\lambda(\lambda)$ at wavelength λ, aperture area, or detector response to a known incident power. A radiometric measurement can help determine the value of one of the above quantities if enough of the others are known. However, one must often account for diffraction of radiation by the aperture, and the effects of diffraction on radiometric measurements have been studied by Lommel,[1] Wolf,[2] Focke,[3] Blevin,[4] Steel *et al.*,[5] Boivin,[6] Shirley,[7] Edwards and McCall,[8] and others.

In this work, diffraction effects on spectral power reaching the detector and total power reaching the detector in the case of a Planckian source are considered anew. This is done within scalar diffraction theory in the paraxial Fresnel approximation. The starting point is a formula derived by Wolf for diffraction effects on spectral power in combination with a recently developed scheme to treat broadband radiation.[9] In a wide range of geometries specified in Section 2, either asymptotic formulas or simple numerical integrals, both of which are derived here, should permit a complete, efficient characterization of diffraction effects for all wavelengths or source temperatures, as appropriate. Asymptotic properties of diffraction effects at small λ or high temperature T are determined to several orders in a small parameter proportional to λ or $1/T$.

2. BACKGROUND

If the dimensions of optical elements are as indicated in Fig. 1 and f is the focal length of the lens, with an aperture corresponding to $f = \pm\infty$, one may first introduce the parameters

$$\begin{aligned} u &= (2\pi R_a{}^2/\lambda)(1/d_s + 1/d_d - 1/f), \\ v_s &= (2\pi/\lambda)(R_s R_a/d_s), \\ v_d &= (2\pi/\lambda)(R_d R_a/d_d), \\ v_0 &= \max(v_s, v_d), \\ \sigma &= \min(v_s, v_d)/\max(v_s, v_d). \end{aligned} \tag{1}$$

For the case of a point source located in the source plane on the optical axis, which corresponds to the limit $R_s \to 0$, implying $v_0 = v_d$ and $\sigma = 0$, Wolf provides a convenient expression for the fraction of spectral power passing through the aperture or lens that lands on the detector, denoted $L(u, v)$, with $v = v_0 = v_d$. Introducing

$$Q_{2s}(v) = \sum_{p=0}^{2s} (-1)^p [J_p(v) J_{2s-p}(v) + J_{p+1}(v) J_{2s+1-p}(v)] \tag{2}$$

and

$$Y_n(u, v) = \sum_{s=0}^{\infty} (-1)^s (n + 2s)(v/u)^{n+2s} J_{n+2s}(v), \tag{3}$$

Wolf provides two equivalent expressions for $L(u, v)$. One expression is convenient for $|v/u| < 1$, and the other expression is convenient for $|v/u| > 1$. If one introduces the additional functions

$$L_B(v, w) = \sum_{s=0}^{\infty} (-1)^s w^{2s} Q_{2s}(v)/(2s + 1) \tag{4}$$

and, with $g = (w + 1/w)/2$,

$$L_X(v, w) = (4w/v)[Y_1(v/w, v)\cos(gv) + Y_2(v/w, v)\sin(gv)], \tag{5}$$

Wolf shows that $L(u, v) = 1 - L_B(v, w)$, with $w = u/v$ for $|v/u| > 1$, and $L(u, v) = w^2[1 + L_B(v, w)] - L_X(v, w)$, with $w = v/u$ for $|v/u| < 1$. When $L(u, v)$ is so expressed as a function of v and w, the right-hand side depends on λ only through v, and not through w, which does not depend on λ.

For the case of an extended source, one can relate the spectral power $\Phi_\lambda(\lambda)$ that reaches the detector to source

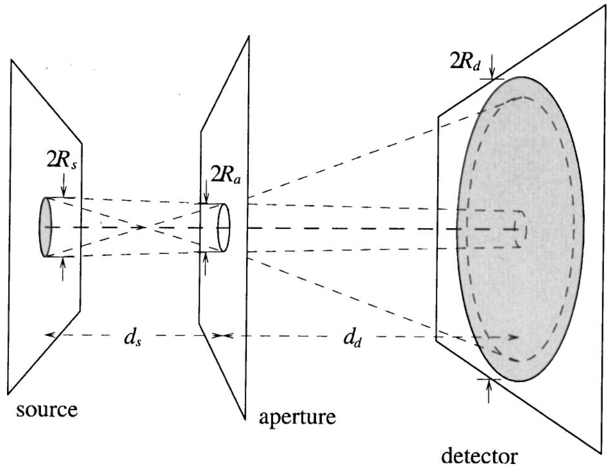

Fig. 1. Class of optical setup considered in this work. The aperture can be limiting (as shown in this case) or nonlimiting.

spectral radiance through an expression derived by Shirley[7] and related to Wolf's result by Edwards and McCall.[8] This gives

$$\Phi_\lambda(\lambda) = C\int_{-1}^{1} \mathrm{d}x(1+\sigma x)^{-1}\{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2} L(u, v_0(1+\sigma x))L_\lambda(\lambda), \tag{6}$$

with $C = 4\pi^4 R_a{}^4 R_s{}^2 R_d{}^2/[d_s{}^2 d_d{}^2(\lambda v_0)^2]$. Henceforth, the argument $v_0(1+\sigma x)$ may be abbreviated v or α/λ. Note that α is independent of λ, and that dependence of α on the setup geometry and x is implicit. We now make the restriction in this work that geometries considered should never involve a value of w that is too close to unity. This means that an overfilled detector or oversampled source should be well overfilled or oversampled, respectively, and that an underfilled detector or undersampled source should be well underfilled or well undersampled, respectively. As an example, in Fig. 1 the detector perimeter should be well outside the larger dashed circle in the detector plane. However, it should be well inside the smaller dashed circle in the case of a nonlimiting aperture.

Because we have $v_0 \propto \lambda^{-1}$, C is independent of λ. Therefore the only dependence of the right-hand side on λ occurs because of $L(u, v)$ and $L_\lambda(\lambda)$. Furthermore, just as in the case of a point source, because we have either

$$\begin{aligned} L(u, v_0(1+\sigma x)) &= L(u, v) = 1 - L_B(v, w) \\ &= 1 - L_B(\alpha/\lambda, w) \end{aligned} \tag{7}$$

or

$$\begin{aligned} L(u, v_0(1+\sigma x)) &= L(u, v) \\ &= w^2[1 + L_B(v, w)] - L_X(v, w) \\ &= w^2[1 + L_B(\alpha/\lambda, w)] - L_X(\alpha/\lambda, w), \end{aligned} \tag{8}$$

the dependence of the rightmost expressions for $L(u, v_0(1+\sigma x))$ on λ arises through $v = \alpha/\lambda$ alone, and not through w.

For a thermal source with emissivity ϵ, the total power reaching the detector is

$$\Phi = C\int_{-1}^{1} \mathrm{d}x(1+\sigma x)^{-1}\{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2} \int_0^\infty \mathrm{d}\lambda L(u, v)L_\lambda(\lambda), \tag{9}$$

where one has

$$L_\lambda(\lambda) = \frac{\epsilon c_1}{\pi\lambda^5}\left[\exp\left(\frac{c_2}{\lambda T}\right) - 1\right]^{-1}. \tag{10}$$

If we introduce the parameter

$$A = c_2/(\alpha T) \tag{11}$$

and change the variable of integration from λ to v, we have either

$$\begin{aligned} \int_0^\infty \mathrm{d}\lambda L(u, v)L_\lambda(\lambda) &= \frac{\epsilon c_1}{\pi\alpha^4}\int_0^\infty \frac{\mathrm{d}v v^3}{\exp(Av) - 1} \times [1 - L_B(v, w)] \\ &= \frac{\epsilon c_1}{\pi\alpha^4}\left[\frac{6\zeta(4)}{A^4} - F_B(A, w)\right] \end{aligned} \tag{12}$$

or

$$\begin{aligned} \int_0^\infty \mathrm{d}\lambda L(u, v)L_\lambda(\lambda) &= \frac{\epsilon c_1}{\pi\alpha^4}\int_0^\infty \frac{\mathrm{d}v v^3}{\exp(Av) - 1}\{w^2[1 + L_B(v, w)] - L_X(v, w)\} \\ &= \frac{\epsilon c_1}{\pi\alpha^4}\left[\frac{6w^2\zeta(4)}{A^4} + w^2 F_B(A, w) - F_X(A, w)\right]. \end{aligned} \tag{13}$$

Here we have introduced the functions

$$F_B(A, w) = \int_0^\infty \frac{\mathrm{d}v v^3}{\exp(Av) - 1} L_B(v, w), \tag{14}$$

$$F_X(A, w) = \int_0^\infty \frac{\mathrm{d}v v^3}{\exp(Av) - 1} L_X(v, w), \tag{15}$$

as well as the Riemann zeta function $\zeta(z) = \Sigma_{n=1}^{\infty} n^{-z}$ for $z > 1$.

The above developments provide a way to predict spectral or total power reaching the detector with diffraction effects taken into account. Neglect of diffraction effects amounts to setting $L_B(v, w) = L_X(v, w) = 0$ or $F_B(A, w) = F_X(A, w) = 0$.

3. EVALUATION OF $L_B(v, w)$ AND $F_B(A, w)$

From the integral representation of a Bessel function,

$$J_m(v) = \frac{(-i)^m}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \exp(ivx + im\theta), \tag{16}$$

where we use the shorthand $x = \cos\theta$, we have

$$\begin{aligned}\sum_{p=0}^{2s} & (-1)^p J_p(v) J_{2s-p}(v) \\ &= (-1)^s \sum_{p=-s}^{s} (-1)^p J_{s+p}(v) J_{s-p}(v) \\ &= (-1)^s \sum_{p=-s}^{s} (-1)^p \left\{ \frac{i^{-s-p}}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \exp(ivx) \right. \\ &\quad \left. \times \exp[i(s+p)\theta] \right\} \\ &\quad \times \left\{ \frac{i^{-s+p}}{2\pi} \int_0^{2\pi} \mathrm{d}\theta' \exp(ivx') \exp[i(s-p)\theta'] \right\}. \quad (17)\end{aligned}$$

For real v, $J_{s-p}(v)$ is real. One may therefore replace $+i$ with $-i$ everywhere in the related integral representation. Simplification gives

$$\begin{aligned}\sum_{p=0}^{2s} & (-1)^p J_p(v) J_{2s-p}(v) \\ &= (-1)^s \sum_{p=-s}^{s} (-1)^p \left\{ \frac{i^{-s-p}}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \right. \\ &\quad \left. \times \exp[i(vx + s\theta + p\theta)] \right\} \\ &\quad \times \left\{ \frac{i^{s-p}}{2\pi} \int_0^{2\pi} \mathrm{d}\theta' \exp[i(-vx' - s\theta' + p\theta')] \right\} \\ &= \frac{(-1)^s}{(2\pi)^2} \sum_{p=-s}^{s} \int_0^{2\pi} \mathrm{d}\theta \exp[i(vx + s\theta + p\theta)] \\ &\quad \times \int_0^{2\pi} \mathrm{d}\theta' \exp[i(-vx' - s\theta' + p\theta')] \\ &= \frac{(-1)^s}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\theta \exp(ivx) \int_0^{2\pi} \mathrm{d}\theta' \exp(-ivx') \\ &\quad \times \eta^s \left(\frac{h^{2s+1} - h^{-2s-1}}{h - h^{-1}} \right) \quad (18)\end{aligned}$$

with $h = \exp[i(\theta + \theta')/2]$ and $\eta = \exp[i(\theta - \theta')]$. Similar analysis gives

$$\begin{aligned}\sum_{p=0}^{2s} & (-1)^p J_{p+1}(v) J_{2s+1-p}(v) \\ &= \frac{(-1)^s}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\theta \exp(ivx) \int_0^{2\pi} \mathrm{d}\theta' \exp(-ivx') \\ &\quad \times \eta^{s+1} \left(\frac{h^{2s+1} - h^{-2s-1}}{h - h^{-1}} \right), \quad (19)\end{aligned}$$

from which one may deduce

$$\begin{aligned}Q_{2s}(v) &= \frac{(-1)^s}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\theta \exp(ivx) \int_0^{2\pi} \mathrm{d}\theta' \exp(-ivx') \\ &\quad \times \left(\frac{1+\eta}{h - h^{-1}} \right) [h(\eta h^2)^s - h^{-1}(\eta h^{-2})^s]. \quad (20)\end{aligned}$$

Introducing $w_1 = w \exp(i\theta)$ and $w_2 = w \exp(-i\theta')$ we have

$$\begin{aligned}L_B(v, w) &= \sum_{s=0}^{\infty} (-1)^s w^{2s} Q_{2s}(v)/(2s+1) \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\theta \exp(ivx) \int_0^{2\pi} \mathrm{d}\theta' \exp(-ivx') \\ &\quad \times \left(\frac{1+\eta}{h - h^{-1}} \right) \left(h \sum_{s=0}^{\infty} \frac{w_1^{2s}}{2s+1} \right. \\ &\quad \left. - h^{-1} \sum_{s=0}^{\infty} \frac{w_2^{2s}}{2s+1} \right) \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\theta \exp(ivx) \int_0^{2\pi} \mathrm{d}\theta' \exp(-ivx') \\ &\quad \times \left(\frac{1+\eta}{h - h^{-1}} \right) \left[\frac{h}{2w_1} \log_e \left(\frac{1+w_1}{1-w_1} \right) \right. \\ &\quad \left. - \frac{h^{-1}}{2w_2} \log_e \left(\frac{1+w_2}{1-w_2} \right) \right]. \quad (21)\end{aligned}$$

Here one uses a version of the log function that has the indicated series expansion. One way to do this is to have $\log_e 1 = 0$ and have the branch cut on the negative real axis. For $F_B(A, w)$ one has

$$\begin{aligned}F_B(A, w) &= \int_0^{\infty} \frac{\mathrm{d}v v^3}{\exp(Av) - 1} L_B(v, w) \\ &= \frac{1}{2} \int_0^{\infty} \frac{\mathrm{d}v v^3}{\exp(Av) - 1} [L_B(v, w) + L_B(-v, w)] \\ &= \frac{3}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\theta' \left(\frac{1+\eta}{h - h^{-1}} \right) \\ &\quad \times \left[\frac{h}{2w_1} \log_e \left(\frac{1+w_1}{1-w_1} \right) \right. \\ &\quad \left. - \frac{h^{-1}}{2w_2} \log_e \left(\frac{1+w_2}{1-w_2} \right) \right] S(A, x - x'), \quad (22)\end{aligned}$$

with

$$S(x, y) = S(x, -y) = \sum_{n=1}^{\infty} [(nx + iy)^{-4} + (nx - iy)^{-4}]. \tag{23}$$

$S(x, y)$ and methods to evaluate it are discussed elsewhere.[9] A short summary is as follows. For positive x and y, one can introduce $z = 2\pi y/x$ and $f = 1/[1 - \exp(-z)]$. One has

$$S(x, y) = -(2\pi/x)^4[1/z^4 + (f - 7f^2 + 12f^3 - 6f^4)/6]. \tag{24}$$

This is useful except when $z \ll 1$, when one may use

$$S(x, y) = 32\pi^4/x^4[1/1440 - z^2/6048 + z^4/69120 - \cdots]. \tag{25}$$

At small v or large A, the above integrals may be evaluated numerically. However, asymptotic expansions for the integrals are desirable at large v or small A, as described below. Obtaining the same results at intermediate values of v or A by both methods helps ensure efficacy of asymptotic expansions and numerical integration at extreme values of v or A.

A. Evaluation of $L_B(v, w)$ at Large v

For large positive v, evaluation of $Q_{2s}(v)$ can be simplified by using an asymptotic expression for Bessel functions of large nonnegative argument[10]:

$$J_m(v) \sim (2/\pi v)^{1/2}\left[\cos\zeta \sum_{s=0}^{\infty} (-1)^s A_{2s}(m) v^{-2s} - \sin\zeta \sum_{s=0}^{\infty} (-1)^s A_{2s+1}(m) v^{-(2s+1)}\right], \tag{26}$$

with $\zeta = v - m\pi/2 - \pi/4$, $A_0(m) = 1$, and $A_s(m) = (4m^2 - 1^2)(4m^2 - 3^2)\ldots[4m^2 - (2s-1)^2]/[8^s(s!)]$ for all other s. When using this expression for $J_m(v)$ in $Q_{2s}(v)$ one obtains terms divided by increasing powers of v. If one collects and simplifies the result, one can obtain

$$\begin{aligned} Q_{2s}(v) = (-1)^s(2s+1)\Bigg[&\frac{2}{\pi v} - \frac{\cos 2v}{\pi v^2} - \frac{16s^4 + 32s^3 + 8s^2 - 8s - 3}{12\pi v^3} + \left(\frac{8s^2 + 8s - 1}{4\pi v^3}\right)\sin(2v) \\ &+ \left(\frac{64s^4 + 128s^3 - 16s^2 - 80s + 9}{32\pi v^4}\right)\cos(2v)\Bigg] + O(v^{-5}). \end{aligned} \tag{27}$$

To sum over s to obtain $L_B(v, w)$, it is helpful to introduce the shorthand

$$\sigma_k = \sum_{s=0}^{\infty} s^k w^{2s} = \left[w^2 \frac{\mathrm{d}}{\mathrm{d}(w^2)}\right]^k \frac{1}{1 - w^2} = \frac{W_k(w^2)}{(1 - w^2)^{k+1}}, \tag{28}$$

where the first few W_k polynomials are

$$W_0(x) = 1, \quad W_3(x) = x^3 + 4x^2 + x,$$

$$W_1(x) = x, \quad W_4(x) = x^4 + 11x^3 + 11x^2 + x,$$

$$W_2(x) = x^2 + x,$$

$$W_5(x) = x^5 + 26x^4 + 66x^3 + 26x^2 + x. \tag{29}$$

This gives

$$\begin{aligned} L_B(v, w) &= \sum_{s=0}^{\infty} (-1)^s w^{2s} Q_{2s}(v)/(2s+1) \\ &= \Bigg[\frac{2\sigma_0}{\pi v} - \frac{\sigma_0 \cos(2v)}{\pi v^2} - \frac{16\sigma_4 + 32\sigma_3 + 8\sigma_2 - 8\sigma_1 - 3\sigma_0}{12\pi v^3} \\ &\quad + \left(\frac{8\sigma_2 + 8\sigma_1 - \sigma_0}{4\pi v^3}\right)\sin(2v) + \left(\frac{64\sigma_4 + 128\sigma_3 - 16\sigma_2 - 80\sigma_1 + 9\sigma_0}{32\pi v^4}\right) \\ &\quad \times \cos(2v)\Bigg] + O(v^{-5}). \end{aligned} \tag{30}$$

B. Evaluation of $F_B(A, w)$ at Small A

At small A, one may first find the analytic properties of the following as a function of A and s:

$$I_{2s}(A) = \int_0^{\infty} \frac{\mathrm{d}v v^3}{\exp(Av) - 1} Q_{2s}(v) = \sum_{n=0}^{\infty} i_{2s}(nA), \tag{31}$$

with

$$i_{2s}(A) = \int_0^{\infty} \mathrm{d}v v^3 \exp(-Av) Q_{2s}(v), \tag{32}$$

and then evaluate $F_B(A, w) = \Sigma_{s=0}^{\infty}(-1)^s w^{2s} I_{2s}(A)/(2s+1)$. Because this sum involves a geometric series in w that is always less than unity, one can sum over s numerically. One can also sum contributions at various orders in A over s analytically. Many of the resulting analytical expressions are lengthy, and we do not discuss them further here. Numerical summation over s is less practical when w is very close to unity.

From the series expansions of Bessel functions of nonnegative integer orders a and b, we have

$$J_a(v)J_b(v) = \left(\frac{v}{2}\right)^{a+b} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-v^2/4)^{r+s}}{r!s!(r+a)!(s+b)!}$$

$$= \left(\frac{v}{2}\right)^{a+b} \sum_{m=0}^{\infty} \left(-\frac{v^2}{4}\right)^m \sum_{k=0}^{m} [k!(m-k)!(k + a)!(m+b-k)!]^{-1}$$

$$= \left(\frac{v}{2}\right)^{a+b} \sum_{m=0}^{\infty} \left(-\frac{v^2}{4}\right)^m \sum_{k=0}^{m} \{k!(m+b - k)!(m-k)![m+a-(m-k)]!\}^{-1}$$

$$= \left(\frac{v}{2}\right)^{a+b} \sum_{m=0}^{\infty} \left(-\frac{v^2}{4}\right)^m [(m+a)!(m + b)!]^{-1} \sum_{k=0}^{m} \binom{m+b}{k}\binom{m+a}{m-k}$$

$$= \left(\frac{v}{2}\right)^{a+b} \sum_{m=0}^{\infty} \left(-\frac{v^2}{4}\right)^m [(m+a)!(m + b)!]^{-1} \binom{2m+a+b}{m}. \tag{33}$$

Therefore the integral

$$\int_0^{\infty} \mathrm{d}v v^{3+a+b+2m} \exp(-Av) = \frac{\Gamma(2m+a+b+4)}{A^{2m+a+b+4}} \tag{34}$$

implies that we have

$$\int_0^{\infty} dv v^3 \exp(-Av) J_a(v) J_b(v) = \frac{1}{2^{a+b}A^{a+b+4}} \sum_{m=0}^{\infty} \left(-\frac{1}{4A^2}\right)^m \times \frac{\Gamma(2m+a+b+4)\Gamma(2m+a+b+1)}{(m+a)!(m+b)!m!\Gamma(m+a+b+1)}. \tag{35}$$

Generalizing this to Wolf's result, we have

$$i_{2s}(A) = \sum_{c=0}^{1} \frac{1}{2^{2s+2c}A^{2s+2c+4}} \sum_{m=0}^{\infty} \left(-\frac{1}{4A^2}\right)^m \frac{\Gamma(2m+2s+2c+4)\Gamma(2m+2s+2c+1)T(m+c,2s)}{m!\Gamma(m+2s+2c+1)}, \tag{36}$$

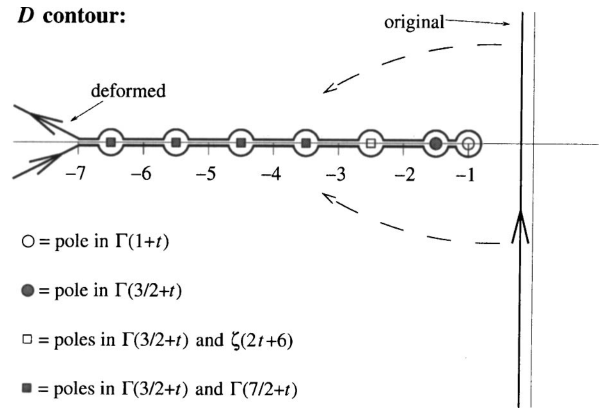

Fig. 2. Contour integration for Barnes's integral for hypergeometric function discussed in text, for $s = 0$.

with

$$T(M, 2s) = \sum_{p=0}^{2s} (-1)^p [(M+p)!(M+2s-p)!]^{-1} = (-1)^s \sum_{p=-s}^{s} (-1)^p [(M+s+p)!(M+s - p)!]^{-1}. \tag{37}$$

From combinatorics, we have $T(0, 2s) = \delta_{s,0}$ and

$$T(M, 2s)|_{M>0} = [(M+s)\Gamma(M)\Gamma(1+M+2s)]^{-1}. \tag{38}$$

We know this, using

$$T(0, 2s) = \frac{1}{(2s)!} \sum_{p=0}^{2s} \binom{2s}{p} (-1)^p = \frac{(1-1)^{2s}}{(2s)!} = \delta_{s,0}, \tag{39}$$

and we may deduce the value of $T(M, 2s)$ for $M > 0$ using

$$T(M, 2s)|_{M>0} = \frac{(-1)^M}{(2M+2s)!} \sum_{p=-s}^{s} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \exp(2ip\theta) \times [\exp(i\theta) - \exp(-i\theta)]^{2M+2s}. \tag{40}$$

Abbreviating $z = (4A^2)^{-1}$, we can expand the sum over c and partition terms with $m + c = 0$ from the others to obtain

$$
\begin{aligned}
i_{2s}(A) = {} & 6A^{-4}\delta_{s,0} + \frac{1}{A^4(2A)^{2s}}\sum_{m=0}^{\infty} z(-z)^m \\
& \times \frac{\Gamma(2m+2s+6)\Gamma(2m+2s+3)}{m!\Gamma(m+2s+3)(m+1+s)\Gamma(m+1)\Gamma(m+2s+2)} \\
& + \frac{1}{A^4(2s)^{2s}}\sum_{m=0}^{\infty}(-z)^{m+1}\frac{\Gamma(2m+2s+6)\Gamma(2m+2s+3)}{(m+1)!\Gamma(m+2s+2)(m+1+s)\Gamma(m+1)\Gamma(m+2s+2)},
\end{aligned} \tag{41}
$$

which simplifies to

$$
\begin{aligned}
i_{2s}(A) &= 6A^{-4}\delta_{s,0} + \frac{z}{A^4(2A)^{2s}}\sum_{m=0}^{\infty}\frac{(-z)^m}{m!}\frac{\Gamma(2m+2s+6)\Gamma(2m+2s+3)}{(m+1+s)\Gamma(m+2s+2)}\left[\frac{1}{m!\Gamma(m+2s+3)}\right. \\
&\quad \left. - \frac{1}{(m+1)!\Gamma(m+2s+2)}\right] \\
&= 6A^{-4}\delta_{s,0} + \frac{2}{2^{2s+2}A^{2s+6}}\sum_{m=0}^{\infty}\frac{(-z)^m\Gamma(2m+2s+6)\Gamma(2m+2s+2)}{m!\Gamma(m+2s+2)}\left[\frac{(m+1)-(m+2s+2)}{(m+1)!\Gamma(m+2s+3)}\right] \\
&= 6A^{-4}\delta_{s,0} - \frac{2(2s+1)}{2^{2s+2}A^{2s+6}}\sum_{m=0}^{\infty}\left(\frac{(-z)^m}{m!}\right)\frac{\Gamma(2m+2s+6)\Gamma(2m+2s+2)}{\Gamma(m+2)\Gamma(m+2s+2)\Gamma(m+2s+3)}.
\end{aligned} \tag{42}
$$

By using $\Gamma(x+1) = x\Gamma(x)$, we have $\Gamma(x+2m) = \Gamma(x)2^{2m}[(x+1)/2]_m(x/2)_m$ and

$$
i_{2s}(A) = 6A^{-4}\delta_{s,0} - \frac{2(2s+1)\Gamma(2s+6)\Gamma(2s+2)}{2^{2s+2}A^{2s+6}}\sum_{m=0}^{\infty}\left(\frac{(-z)^m}{m!}\right)\frac{2^{4m}(s+3)_m(s+7/2)_m(s+1)_m(s+3/2)_m}{\Gamma(2)\Gamma(2s+2)\Gamma(2s+3)(2)_m(2s+2)_m(2s+3)_m}. \tag{43}
$$

From $2^{4m}(-z)^m = (-4/A^2)^m$, $\Gamma(2) = 1$, and $\Gamma(2s+6)/\Gamma(2s+3) = (2s+5)(2s+4)(2s+3)$, we have

$$
\begin{aligned}
i^{2s}(A) &= 6A^{-4}\delta_{s,0} - \frac{2(2s+1)(2s+3)(2s+4)(2s+5)}{2^{2s+2}A^{2s+6}}\sum_{m=0}^{\infty}\left(\frac{(-4/A^2)^m}{m!}\right)\frac{(s+3)_m(s+7/2)_m(s+1)_m(s+3/2)_m}{(2)_m(2s+2)_m(2s+3)_m} \\
&= 6A^{-4}\delta_{s,0} - \left(\frac{2(2s+1)(2s+3)(2s+4)(2s+5)}{2^{2s+2}A^{2s+6}}\right) \\
&\quad \times {}_4F_3(s+1, s+3/2, s+3, s+7/2; 2, 2s+2, 2s+3; -4/A^2).
\end{aligned} \tag{44}
$$

From Barnes integral representation of a generalized hypergeometric function,[11] we have

$$
\begin{aligned}
& {}_4F_3(s+1, s+3/2, s+3, s+7/2; 2, 2s+2, 2s+3; -4/A^2) \\
&\quad = \frac{1}{2\pi i}\frac{\Gamma(2)\Gamma(2s+2)\Gamma(2s+3)}{\Gamma(s+1)\Gamma(s+3/2)\Gamma(s+3)\Gamma(s+7/2)} \\
&\qquad \times \int_D \mathrm{d}t\,\frac{\Gamma(s+1+t)\Gamma(s+3/2+t)\Gamma(s+3+t)\Gamma(s+7/2+t)\Gamma(-t)\exp(\lambda t)}{\Gamma(2+t)\Gamma(2s+2+t)\Gamma(2s+3+t)}.
\end{aligned} \tag{45}
$$

We have introduced the parameter $\lambda = \log_e(4/A^2)$ to express $(4/A^2)^t$ as an exponential. The contour D in the complex-t plane should run from $-i\infty$ to $+i\infty$ and pass to the left of all poles in $\Gamma(-t)$ but to the right of all other poles in the integrand. If we note

$$
\begin{aligned}
\frac{2(2s+1)(2s+3)(2s+4)(2s+5)\Gamma(2s+2)\Gamma(2s+3)}{2^{2s+2}A^{2s+6}\Gamma(s+1)\Gamma(s+3/2)\Gamma(s+3)\Gamma(s+7/2)} &= \frac{2(2s+1)}{2^{2s+2}A^{2s+6}}\left(\frac{\Gamma(2s+2)}{\Gamma(s+1)\Gamma(s+3/2)}\right)\left(\frac{\Gamma(2s+6)}{\Gamma(s+3)\Gamma(s+7/2)}\right) \\
&= \frac{2(2s+1)}{2^{2s+2}A^{2s+6}}\left(\frac{2^{2(s+1)-1}}{\Gamma(1/2)}\right)\left(\frac{2^{2(s+3)-1}}{\Gamma(1/2)}\right) = \frac{2s+1}{2\pi}\left(\frac{2}{A}\right)^{2s+6},
\end{aligned} \tag{46}
$$

the combination of Eq. (44), Eq. (45), and Eq. (46) gives

$$i_{2s}(A) = \frac{6}{A^4}\delta_{s,0} - \frac{2s+1}{2\pi}\left(\frac{2}{A}\right)^{2s+6}\frac{1}{2\pi i}\int_D \mathrm{d}t \frac{\Gamma(s+1+t)\Gamma(s+3/2+t)\Gamma(s+3+t)\Gamma(s+7/2+t)\Gamma(-t)\exp(\lambda t)}{\Gamma(2+t)\Gamma(2s+2+t)\Gamma(2s+3+t)}. \tag{47}$$

By initially placing the contour D just to the left of the imaginary axis, we may also deduce

$$I_{2s}(A) = \frac{6\zeta(4)\delta_{s,0}}{A^4} - \frac{(2s+1)4^{s+3}}{4\pi^2 i A^{2s+6}}\int_D \mathrm{d}t \frac{\Gamma(s+1+t)\Gamma(s+3/2+t)\Gamma(s+3+t)\Gamma(s+7/2+t)\Gamma(-t)\zeta(2s+2t+6)\exp(\lambda t)}{\Gamma(2+t)\Gamma(2s+2+t)\Gamma(2s+3+t)}. \tag{48}$$

This uses $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ for $\mathrm{Re}\, z > 1$.

Analytic continuation of the integrand and deformation of D leads to a term-by-term series expansion of $I_{2s}(A)$ in increasing powers of A and powers of A multiplied by $\log_e A$. Because of this, we have a result of the form

$$I_{2s}(A) \sim \sum_{p=-4}^{\infty} (C_{s,p} + L_{s,p} \log_e A) A^p. \tag{49}$$

The first few terms of this expansion give a very accurate result at sufficiently small A. This complements the double numerical integration described earlier, which is increasingly difficult at small A. The strategy used is as follows: One tries to move the entire contour D to the left as far as possible. As this is done, poles are encountered on the real axis and nowhere else in the left half-plane. The contour must cross the real axis to the right of these poles. Near the real axis, D can take a detour that runs just below the real axis to the right, circles the right-most pole in counterclockwise fashion, and runs just above the real axis to the left. Figure 2 illustrates this for $s = 0$. In the limit of small A, contributions to the integral at increasing orders in A arise from circling poles on the real axis in counterclockwise fashion. The residues determine the expansion coefficients.

The integrand contains many factors in the numerator and denominator, several of which can have poles and–or zeros. These factors "conspire" to form an overall pole structure of the integrand. One or more poles in these factors coincide if they occur at the same value of t. The denominator of the integrand has no zeros, so it does not contribute in this way to the pole structure of the integrand. However, poles in the denominator coincident with poles in the numerator can negate or modify the latter's effects. Poles in the numerator occur at certain integer and half-integer values of t. Up to two poles can coincide in the factors in the numerator without being canceled by poles in the factors in the denominator. Therefore, the relevant pole structure of the integrand is fully determined if one expands every factor to at least one order beyond lowest order about the value of t at which any factor is singular. One has

$$\Gamma(z+\epsilon) = \Gamma(z)[1+\epsilon\psi(z)] + O(\epsilon^2),$$

except near $z = -N$, $N = 0, 1, 2,...$, where one has

$$\Gamma(z+\epsilon) = \Gamma(-N+\epsilon) = [(-1)^N/N!][1/\epsilon + \psi(N+1)] + O(\epsilon).$$

One has

$$\zeta(z+\epsilon) = \zeta(z) + \epsilon\zeta'(z) + O(\epsilon^2),$$

except near $z = 1$, where one has

$$\zeta(z+\epsilon) = \zeta(1+\epsilon) = 1/\epsilon + \gamma + O(\epsilon).$$

Finally, one has

$$\exp[\lambda(z+\epsilon)] = [1+\epsilon\lambda + O(\epsilon^2)]\exp(\lambda z)$$

everywhere.

Poles at integer t, which correspond to even values of p, can arise from the factors $\Gamma(s+1+t)$ and $\Gamma(s+3+t)$. For $s = 0$, $\Gamma(s+1+t)$ and the integrand have a simple pole at $t = -1$, giving a contribution to $I_{2s}(A)$ of

$$-\frac{1}{2\pi}\left(\frac{2}{A}\right)^6\left(\frac{\Gamma(1/2)\Gamma(2)\Gamma(5/2)\Gamma(1)\zeta(4)(4/A^2)^{-1}}{\Gamma(1)\Gamma(1)\Gamma(2)}\right) = -\frac{6\zeta(4)}{A^4}. \tag{50}$$

For $s > 0$, the pole at $s+1+t = 0$ has no effect at $t = -1$ because it coincides with a pole in the denominator, and the integrand is regular. Thus the leading term in Eq. (48) is exactly canceled, and we have $C_{s,-4} = L_{s,-4} = 0$ for all s.

If, and only if, we have $-2s-2 < t \leqslant -s-3$, there is one more pole in the factors $\Gamma(s+1+t)$ and $\Gamma(s+3+t)$ than in the factors in the denominator. The inequality can be fulfilled only for $s > 1$. However, the Riemann zeta function crosses zero at all negative, even integers, so the only other pole of the integrand at integer t occurs at $t = -s-3$ for $s > 1$, for which one has $\zeta(2s+2t+6) = \zeta(0) = -1/2$. There, one has a contribution to $I_{2s}(A)$ with

$$C_{s,0} = (-1)^{s+1}(2s+1)(s^6 + 3s^5 + s^4 - 3s^3 - 2s^2)/6. \tag{51}$$

(This expression is automatically zero for $s \leqslant 1$.) Other than $C_{s,0}$ for $s > 1$, we have $C_{s,p} = 0$ and $L_{s,p} = 0$ for all even p.

Poles at half-integer t, which correspond to odd values of p, can arise from the factors, $\Gamma(s+3/2+t)$, $\Gamma(s+7/2+t)$, and $\zeta(2s+2t+6)$. At $t = -s-3/2$, only $\Gamma(s+3/2+t)$ has a pole, which gives $C_{s,-3} = (-1)^s(2s+1)[4\zeta(3)/\pi]$ and $L_{s,-3} = 0$. For this term, direct summation over s can be done analytically to yield the compact expression

$$\sum_{s=0}^{\infty} (-1)^s w^{2s} C_{s,-3} A^{-3}/(2s+1) = 4\zeta(3)A^{-3}/[\pi(1-w^2)],$$

which determines the leading-order effects of diffraction and illustrates the breakdown of the present asymptotic expansion as w approaches unity.

At $t = -s - 5/2$, $\Gamma(s + 3/2 + t)$ and $\zeta(2s + 2t + 6)$ have simple poles that coincide. By expanding all factors in the integrand to one order above lowest order, the integrand is found to have a second-order pole, a first-order pole, and an analytic part. The residue of the first-order pole leads to

$$C_{s,-1} = -\frac{(2s+3)(2s+1)^3(2s-1)(-1)^s}{24\pi} \times \left[-2\gamma - 2\log_e 2 + \frac{11}{3} - \psi\left(s+\frac{5}{2}\right) - \psi\left(s+\frac{3}{2}\right) - \psi\left(s+\frac{1}{2}\right) - \psi\left(s-\frac{1}{2}\right)\right] \tag{52}$$

and

$$L_{s,-1} = +\frac{(2s+3)(2s+1)^3(2s-1)(-1)^s}{12\pi}. \tag{53}$$

The identity $\psi(1/2 + s) = \psi(1/2 - s)$ for integer s was used to set the coefficient of s to $+1$ in the arguments of all digamma functions, simplifying summation of $C_{s,-1}$ over s.

At $t = -s - 7/2 - \tau$, with $\tau = 0,1,2,\ldots$, both $\Gamma(s + 3/2 + t)$ and $\Gamma(s + 7/2 + t)$ have simple poles. By a process just like that above, we may deduce all remaining terms. After simplifying and using the reflection formulas $\Gamma(z)\Gamma(-z) = -\pi/[z\sin(\pi z)]$[12] and $\zeta(1-z) = 2(2\pi)^{-z}\Gamma(z)\cos(\pi z/2)\zeta(z)$,[13] as well as the doubling formula $2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \Gamma(1/2)\Gamma(2z)$,[14] the following result can be obtained. If we introduce a prefactor that does not depend on s,

$$P_\tau = \frac{(-1)^{\tau+1}\zeta(-1-2\tau)\Gamma(-5/2-\tau)\Gamma(-1/2-\tau)}{(2\pi)^2 2^{2\tau}\tau!(2+\tau)!} = \frac{8\zeta(2+2\tau)}{\pi^{3+2\tau}\Gamma(6+2\tau)}, \tag{54}$$

and a further prefactor

$$Q_{s,\tau} = (-1)^s(2s+1)\left[\frac{\Gamma(s+7/2+\tau)\Gamma(s+5/2+\tau)}{\Gamma(s-1/2-\tau)\Gamma(s-3/2-\tau)}\right] = \frac{(-1)^s(2s+1)}{2^{8+4\tau}}(2s+2\tau+5)[(2s+2\tau+3)\ldots \times (2s-2\tau-1)]^2(2s-2\tau-3), \tag{55}$$

we may write

$$C_{s,1+2\tau} = P_\tau Q_{s,\tau}[\psi(-5/2-\tau) + \psi(-1/2-\tau) + \psi(3+\tau) + \psi(1+\tau) + 2\log_e 2 + 2\zeta'(-1-2\tau)/\zeta(-1-2\tau) - \psi(s+7/2+\tau) - \psi(s+5/2+\tau) - \psi(s-1/2-\tau) - \psi(s-3/2-\tau)] \tag{56}$$

and $L_{s,1+2\tau} = -2P_\tau Q_{s,\tau}$. Here the logarithmic derivative of the Riemann zeta function may be found by taking the logarithmic derivative of the reflection formula, rearranging to get

$$\zeta'(1-z)/\zeta(1-z) = \log_e 2\pi - \psi(z) + (\pi/2)\tan(\pi z/2) - \zeta'(z)/\zeta(z), \tag{57}$$

and setting $z = 2 + 2\tau$. For convenience and as a reference, the values and derivatives of the Riemann zeta function are given in Table 1 for several even positive integers, and the values of $C_{s,p}$ and $L_{s,p}$ are given in Tables 2 and 3 for low values of s and p.

4. EVALUATION OF $L_X(v, w)$ AND $F_X(A, w)$

Defining $g = (w + 1/w)/2$, we have

$$2[Y_1(v/w, v)\cos(gv) + Y_2(v/w, v)\sin(gv)] = \exp(-igv)\left[\sum_{s=0}^{\infty}(-1)^s w^{1+2s}(1+2s)J_{1+2s}(v) + i\sum_{s=0}^{\infty}(-1)^s w^{2+2s}(2+2s)J_{2+2s}(v)\right] + \text{c.c.} \tag{58}$$

Here c.c. denotes complex conjugate. Using $2mJ_m(v)/v = J_{m-1}(v) + J_{m+1}(v)$, we have

Table 1. Values and Derivatives of Riemann Zeta Function for Lowest Even Positive Integers

z	$\zeta(z)$	$\zeta'(z)$
2	$1.64493407 \times 10^{+00}$	$-9.37548254 \times 10^{-01}$
4	$1.08232323 \times 10^{+00}$	$-6.89112659 \times 10^{-02}$
6	$1.01734306 \times 10^{+00}$	$-1.28521651 \times 10^{-02}$
8	$1.00407736 \times 10^{+00}$	$-2.90195255 \times 10^{-03}$
10	$1.00099458 \times 10^{+00}$	$-6.97033008 \times 10^{-04}$
12	$1.00024609 \times 10^{+00}$	$-1.71382846 \times 10^{-04}$
14	$1.00006125 \times 10^{+00}$	$-4.25414934 \times 10^{-05}$
16	$1.00001528 \times 10^{+00}$	$-1.06024420 \times 10^{-05}$
18	$1.00000382 \times 10^{+00}$	$-2.64700298 \times 10^{-06}$
20	$1.00000095 \times 10^{+00}$	$-6.61353021 \times 10^{-07}$
22	$1.00000024 \times 10^{+00}$	$-1.65294254 \times 10^{-07}$

Table 2. Nontrivial Values of $C_{s,p}$ for Small s and p

p	$C_{0,p}$	$C_{1,p}$	$C_{2,p}$	$C_{3,p}$
−3	$+1.53050638 \times 10^{+00}$	$-4.59151915 \times 10^{+00}$	$+7.65253192 \times 10^{+00}$	$-1.07135447 \times 10^{+01}$
−1	$+9.20438559 \times 10^{-02}$	$+2.23211420 \times 10^{+00}$	$+7.33114044 \times 10^{+01}$	$-7.53412869 \times 10^{+02}$
0	$+0.00000000 \times 10^{+00}$	$+0.00000000 \times 10^{+00}$	$-1.20000000 \times 10^{+02}$	$+1.68000000 \times 10^{+03}$
1	$-8.01181727 \times 10^{-03}$	$+4.09726996 \times 10^{-01}$	$+3.95398202 \times 10^{+01}$	$-8.54352726 \times 10^{+02}$
3	$-2.08826878 \times 10^{-04}$	$+3.31508756 \times 10^{-03}$	$-3.34646070 \times 10^{-01}$	$-6.24723505 \times 10^{+01}$
5	$-2.34849116 \times 10^{-05}$	$+2.41611740 \times 10^{-04}$	$-5.58423250 \times 10^{-03}$	$+1.02514511 \times 10^{+00}$
7	$-4.87331641 \times 10^{-06}$	$+4.05463902 \times 10^{-05}$	$-5.34637337 \times 10^{-04}$	$+2.04274104 \times 10^{-02}$
9	$-1.26915661 \times 10^{-06}$	$+9.98218891 \times 10^{-06}$	$-1.04165740 \times 10^{-04}$	$+2.20276108 \times 10^{-03}$
11	$-1.60469958 \times 10^{-08}$	$+1.45437724 \times 10^{-06}$	$-2.38254603 \times 10^{-05}$	$+4.50795605 \times 10^{-04}$
13	$+1.01558907 \times 10^{-06}$	$-3.93002162 \times 10^{-06}$	$+7.35485758 \times 10^{-06}$	$+7.21975567 \times 10^{-05}$
15	$+2.92228373 \times 10^{-06}$	$-1.24540385 \times 10^{-05}$	$+4.05647409 \times 10^{-05}$	$-1.30219534 \times 10^{-04}$
17	$+8.04269637 \times 10^{-06}$	$-3.41875692 \times 10^{-05}$	$+1.13805012 \times 10^{-04}$	$-4.38836368 \times 10^{-04}$
19	$+2.45164507 \times 10^{-05}$	$-1.02368062 \times 10^{-04}$	$+3.30642482 \times 10^{-04}$	$-1.24676633 \times 10^{-03}$
21	$+8.54734618 \times 10^{-05}$	$-3.49715595 \times 10^{-04}$	$+1.08609035 \times 10^{-03}$	$-3.88204256 \times 10^{-03}$

$$\begin{aligned}
&2[Y_1(v/w,\, v)\cos(gv) + Y_2(v/w,\, v)\sin(g/v)] \\
&= \frac{v\exp(-igv)}{2}\left\{\sum_{s=0}^{\infty}(-1)^s w^{1+2s}[J_{2s}(v) + J_{2+2s}(v)]\right. \\
&\quad \left. + i\sum_{s=0}^{\infty}(-1)^s w^{2+2s}[J_{1+2s}(v) + J_{3+2s}(v)]\right\} + \text{c.c.} \\
&= \frac{vw\exp(-igv)}{2}\left[\sum_{s=0}^{\infty}(-w^2)^s J_{2s}(v)\right. \\
&\quad + iw\sum_{s=0}^{\infty}(-w^2)^s J_{1+2s}(v) + \sum_{s=0}^{\infty}(-w^2)^s J_{2+2s}(v) \\
&\quad \left. + iw\sum_{s=0}^{\infty}(-w^2)^s J_{3+2s}(v)\right] + \text{c.c.} \\
&= \frac{vw\exp(-igv)}{2}\left[\sum_{s=0}^{\infty}(iw)^s J_s(v)\right. \\
&\quad \left. - \sum_{s=0}^{\infty} i^{s+2} w^s J_{s+2}(v)\right] + \text{c.c.} \\
&= \frac{vw\exp(-igv)}{4\pi}\left[\int_0^{2\pi}\mathrm{d}\theta\left(\frac{1-\exp(2i\theta)}{1-w\exp(i\theta)}\right)\exp(ivx)\right] \\
&\quad + \text{c.c.}
\end{aligned} \tag{59}$$

Here we use the abbreviation $x = \cos\theta$. Because x is an even function of θ, we have

$$\begin{aligned}
&2[Y_1(v/w,\, v)\cos(gv) + Y_2(v/w,\, v)\sin(gv)] \\
&= \frac{vw\exp(-igv)}{8\pi}\left\{\int_0^{2\pi}\mathrm{d}\theta\left[\frac{1-\exp(2i\theta)}{1-w\exp(i\theta)}\right.\right. \\
&\quad \left.\left. + \frac{1-\exp(-2i\theta)}{1-w\exp(-i\theta)}\right]\exp(ivx)\right\} + \text{c.c.} \\
&= \frac{vw}{4\pi}\int_0^{2\pi}\mathrm{d}\theta\left[\frac{1-\exp(2i\theta)}{1-w\exp(i\theta)}\right. \\
&\quad \left. + \frac{1-\exp(-2i\theta)}{1-w\exp(-i\theta)}\right]\cos[v(x-g)].
\end{aligned} \tag{60}$$

Simplification gives

$$2[Y_1(v/w,\, v)\cos(gv) + Y_2(v/w,\, v)\sin(gv)] = \frac{vw}{\pi}\int_0^{2\pi}\mathrm{d}\theta\left(\frac{1-x^2}{1+w^2-2wx}\right)\cos[v(x-g)]. \tag{61}$$

This gives the following result that is amenable to numerical integration except at large v:

$$\begin{aligned}
L_X(v,\, w) &= \frac{2w^2}{\pi}\int_0^{2\pi}\mathrm{d}\theta\frac{(1-x^2)\cos[v(x-g)]}{1+w^2-2wx} \\
&= \frac{4w^2}{\pi}\int_0^{\pi}\mathrm{d}\theta\frac{(1-x^2)\cos[v(x-g)]}{1+w^2-2wx}.
\end{aligned} \tag{62}$$

Including only one exponential in the integration over v to obtain $F_X(A,\, w)$ gives

$$\begin{aligned}
&f_x(A,\, w) \\
&= \frac{w^2}{\pi}\int_0^{\infty}\mathrm{d}v v^3\exp(-Av)\int_0^{2\pi}\mathrm{d}\theta\left(\frac{1-x^2}{1+w^2-2wx}\right) \\
&\quad \times \exp[iv(x-g)] + \text{c.c.} \\
&= \frac{w}{4\pi}\left(-\frac{\mathrm{d}}{\mathrm{d}A}\right)^3\int_C\frac{\mathrm{d}z}{iz}\left[\frac{4-(z+1/z)^2}{(w+1/w)-(z+1/z)}\right] \\
&\quad \times \left[\frac{1}{L-i(z+1/z)/2}\right] + \text{c.c.} \\
&= \frac{w}{2\pi}\left(-\frac{\mathrm{d}}{\mathrm{d}A}\right)^3\int_C\frac{\mathrm{d}z}{z} \\
&\quad \times \frac{(z^2-1)^2}{(z-w)(z-1/w)(z^2+1+2iLz)} + \text{c.c.}
\end{aligned} \tag{63}$$

Here we have introduced $L = A + ig$, and z is integrated on the unit circle C in counterclockwise fashion. Including the entire Planckian gives

$$F_X(A, w) = \frac{6w^2}{\pi}\int_0^{2\pi} \mathrm{d}\theta \frac{(1 - x^2)S(A, g - x)}{1 + w^2 - 2wx}$$
$$= \frac{12w^2}{\pi}\int_0^{\pi} \mathrm{d}\theta \frac{(1 - x^2)S(A, g - x)}{1 + w^2 - 2wx}. \quad (64)$$

This is amenable to numerical integration except at small A.

A. Evaluation of $L_X(v, w)$ at Large v

At large v we can again exploit the asymptotic behavior of Bessel functions as we did for $L_B(v, w)$. Collecting terms that are divided by successive inverse powers of v gives

$$Y_1(v/w, v) = \frac{vw}{2}\left(\frac{2}{\pi v}\right)^{1/2}\left[\left(\frac{2\sigma_0 + 4\sigma_1}{v}\right)\sin(v - \pi/4) + \left(\frac{3\sigma_0 + 22\sigma_1 + 48\sigma_2 + 32\sigma_3}{4v^2}\right)\cos(v - \pi/4) + \left(\frac{15\sigma_0 + 62\sigma_1 - 160\sigma_2 - 960\sigma_3 - 1280\sigma_4 - 512\sigma_5}{64v^3}\right)\sin(v - \pi/4)\right] + O(v^{-7/2}), \quad (65)$$

$$Y_2(v/w, v) = -\frac{v}{2}\left(\frac{2}{\pi v}\right)^{1/2}\left[\left(\frac{4\sigma_1}{v}\right)\sin(v + \pi/4) + \left(\frac{-\sigma_1 + 16\sigma_3}{2v^2}\right)\cos(v + \pi/4) + \left(\frac{-9\sigma_1 + 160\sigma_3 - 256\sigma_5}{32v^3}\right)\sin(v + \pi/4)\right] + O(v^{-7/2}). \quad (66)$$

From this one can deduce $L_X(v, w)$.

B. Evaluation of $F_X(A, w)$ at Small A

If one evaluates the first integral in the last step of Eq. (63) by the residue theorem, three poles contribute to the result. These poles are at $z = 0$, $z = w$, and $z = -iL[1 - (1 + 1/L^2)^{1/2}] = -iL(1 - R)$. R is defined as indicated, and L and R depend implicitly on A. Because we have $g > 1$ we have $|L| > 1$ and $|1/L^2| < 1$, and $1 + 1/L^2$ and R are in a circle in the complex plane within a distance of unity from 1. When we are finding R so that z is within the unit circle, the square root is to be taken that has a positive real part. With this sign of R, z is within the unit circle, and with the other sign it is outside, because $-iL(1 - R)$ and $-iL(1 + R)$ are multiplicative inverses and the former has a smaller absolute magnitude. It next helps to note

$$z^2 + 1 + 2iLz = [z + iL(1 + R)][z + iL(1 - R)]$$
$$= z^2 + 1 + 2z[iA - (w + 1/w)/2]$$
$$= [(z - w)(z - 1/w) + 2iAz]. \quad (67)$$

To determine values of two of the residues we note that

$$\left.\frac{(z^2 - 1)^2}{(z - w)(z - 1/w)[(z - w)(z - 1/w) + 2iAz]}\right|_{z=0} = 1, \quad (68)$$

$$\left.\frac{(z^2 - 1)^2}{z(z - 1/w)[(z - w)(z - 1/w) + 2iAz]}\right|_{z=w}$$
$$= \frac{(w^2 - 1)^2}{w(w - 1/w)(2iAw)} = \frac{w - 1/w}{2iA}. \quad (69)$$

For the remaining residue, note that at $z = -iL(1 - R)$ we have $z^2 - 1 = 2iLRz$. This gives

Table 3. Nontrivial Values of $L_{s,p}$ for Small s and p

p	$L_{0,p}$	$L_{1,p}$	$L_{2,p}$	$L_{3,p}$
−1	$-7.95774715 \times 10^{-02}$	$-3.58098622 \times 10^{+00}$	$+6.96302876 \times 10^{+01}$	$-4.09426091 \times 10^{+02}$
1	$+3.73019398 \times 10^{-03}$	$-1.30556789 \times 10^{-01}$	$-1.37084629 \times 10^{+01}$	$+6.33330985 \times 10^{+02}$
3	$+1.94280936 \times 10^{-04}$	$-2.44793980 \times 10^{-03}$	$+1.34636689 \times 10^{-01}$	$+2.69542651 \times 10^{+01}$
5	$+3.54157957 \times 10^{-05}$	$-3.00528323 \times 10^{-04}$	$+4.77506114 \times 10^{-03}$	$-4.34530564 \times 10^{-01}$
7	$+1.25504726 \times 10^{-05}$	$-8.54627420 \times 10^{-05}$	$+7.93582604 \times 10^{-04}$	$-1.88872660 \times 10^{-02}$
9	$+7.19037493 \times 10^{-06}$	$-4.24885791 \times 10^{-05}$	$+2.86629304 \times 10^{-04}$	$-3.70325060 \times 10^{-03}$
11	$+6.04168808 \times 10^{-06}$	$-3.23209187 \times 10^{-05}$	$+1.75751797 \times 10^{-04}$	$-1.55833260 \times 10^{-03}$
13	$+6.99301077 \times 10^{-06}$	$-3.47498843 \times 10^{-05}$	$+1.61599112 \times 10^{-04}$	$-1.10377091 \times 10^{-03}$
15	$+1.06645149 \times 10^{-05}$	$-5.00604874 \times 10^{-05}$	$+2.06659961 \times 10^{-04}$	$-1.16336551 \times 10^{-03}$
17	$+2.07225179 \times 10^{-05}$	$-9.29626269 \times 10^{-05}$	$+3.49369350 \times 10^{-04}$	$-1.69309762 \times 10^{-03}$
19	$+4.99794028 \times 10^{-05}$	$-2.16076365 \times 10^{-04}$	$+7.52587960 \times 10^{-04}$	$-3.23524283 \times 10^{-03}$
21	$+1.46499030 \times 10^{-04}$	$-6.14204007 \times 10^{-04}$	$+2.00886273 \times 10^{-03}$	$-7.82772332 \times 10^{-03}$

$$\left.\frac{(z^2-1)^2}{z(z-w)(z-1/w)[z+iL(1+R)]}\right|_{z=-iL(1-R)}$$
$$=\left.\frac{-4L^2R^2z^2}{z(-2iAz)(2iLR)}\right|_{z=-iL(1-R)}$$
$$=-LR/A=-(L^2+1)^{1/2}/A. \quad (70)$$

Applying knowledge of the three residues gives

$$f_x(A,w)=iw\left(-\frac{d}{dA}\right)^3\left(1+\frac{w-1/w}{2iA}-\frac{(L^2+1)^{1/2}}{A}\right)$$
$$+\text{ c.c.}$$
$$=\left(-\frac{d}{dA}\right)^3\left(\frac{w(w-1/w)}{A}-\frac{iw}{A}\{[(A+ig)^2\right.$$
$$\left.+1]^{1/2}-[(A-ig)^2+1]^{1/2}\}\right). \quad (71)$$

Using

$$[(A+ig)^2+1]^{1/2}=[(A+ig+i)(A+ig-i)]^{1/2}$$
$$=(A+ig+i)^{1/2}(A+ig-i)^{1/2} \quad (72)$$

and expanding each factor according to

$$(A+ig\pm i)^{1/2}=(A+ig)^{1/2}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k-1/2)}{k!\Gamma(-1/2)}$$
$$\times\left(\frac{\pm i}{A+ig}\right)^k, \quad (73)$$

we obtain an expansion of the form

$$[(A+ig)^2+1]^{1/2}=(A+ig)\sum_{k=0}^{\infty}C_k(A+ig)^{-k}$$
$$=A+ig+\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k$$
$$\times\frac{1}{A+ig}. \quad (74)$$

By symmetry C_k is zero for odd k. Similar analysis gives

$$[(A-ig)^2+1]^{1/2}=(A-ig)\sum_{k=0}^{\infty}C_k(A-ig)^{-k}$$
$$=A-ig+\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k$$
$$\times\frac{1}{A-ig}, \quad (75)$$

which exploits equivalence of even numbers of differentiation with respect to $-ig$ or $+ig$.

Combining the results yields

$$\frac{iw}{A}\{[(A+ig)^2+1]^{1/2}-[(A-ig)^2+1]^{1/2}\}$$
$$=-\frac{2gw}{A}+iw\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k\left(\frac{1}{A(A+ig)}\right.$$
$$\left.-\frac{1}{A(A-ig)}\right)$$
$$=-\frac{2gw}{A}+iw\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k\left(\frac{1}{ig}\right)$$
$$\times\left(\frac{2}{A}-\frac{1}{A+ig}-\frac{1}{A-ig}\right), \quad (76)$$

and therefore

$$f_x(A,w)=6\left\{\frac{w(w-1/w)}{A^4}+\frac{2w}{A^4}\right.$$
$$\times\left[g-i\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k\frac{1}{ig}\right]$$
$$+iw\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k\left(\frac{1}{ig}\right)\left[\frac{1}{(A+ig)^4}\right.$$
$$\left.\left.+\frac{1}{(A-ig)^4}\right]\right\}. \quad (77)$$

From Eq. (74) the second term may be recognized as

$$\frac{2w}{A^4}\left[g-i\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k\frac{1}{ig}\right]$$
$$=-\frac{2iw}{A^4}\left[ig+\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k\frac{1}{ig}\right]$$
$$=-2iw[(ig)^2+1]^{1/2}/A^4$$
$$=2w(g^2-1)^{1/2}/A^4. \quad (78)$$

Ambiguity as to which sign of square root to take is removed by considering the limit of large g. On simplification, the first term and second term cancel, to give

$$f_X(A,w)=6iw\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k\left(\frac{1}{ig}\right)\left(\frac{1}{(A+ig)^4}\right.$$
$$\left.+\frac{1}{(A-ig)^4}\right), \quad (79)$$

which implies

$$F_X(A,w)=6iw\sum_{k=0}^{\infty}\frac{C_{k+2}}{k!}\left(\frac{d}{d(ig)}\right)^k\frac{S(A,g)}{ig}. \quad (80)$$

For small A/g the relation $S(A,g)\sim -g^{-4}+O[\exp(-2\pi g/A)]$ implies that

$$F_X(A, w) \sim -\frac{iw}{4}\left(\frac{\mathrm{d}}{\mathrm{d}g}\right)^4 \sum_{k=0}^{\infty} \frac{C_{k+2}}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}(ig)}\right)^k \frac{1}{ig}$$
$$\sim -\frac{iw}{4}\left(\frac{\mathrm{d}}{\mathrm{d}g}\right)^4 \{[(ig)^2 + 1]^{1/2} - ig\}$$
$$\sim \pm\frac{w}{4}\left(\frac{\mathrm{d}}{\mathrm{d}g}\right)^4 [g^2 - 1]^{1/2}$$
$$\sim -\frac{96w^6(1 + 3w^2 + w^4)}{(1 - w^2)^7}. \tag{81}$$

It is clear which sign of the square root to take, because one should obtain a negative result in the limit of small A/g.

5. EXAMPLE APPLICATION

As an example by which the above results can be tested, we consider the PMO6 absolute radiometer described by Brusa and Fröhlich.[15] This instrument is used in absolute solar radiometry. The radiometer entrance pupil is defined by a 5-mm-diameter aperture, and the entrance pupil is set back 95.4 mm behind an 8.5-mm-diameter, view-limiting aperture that helps reduce effects of unwanted radiation on measurements. There are also other baffles between these two apertures that should not contribute to first-order diffraction effects. Meanwhile, the sun and the distance to the sun are the remaining geometrical parameters of interest. For these one has $R_a = 4.25$ mm, $R_d = 2.5$ mm, $d_d = 95.4$ mm, $d_s \approx 1.5 \times 10^{14}$ mm, and $R_s \approx 6.75 \times 10^{11}$ mm.

Suppose we assume a solar surface temperature of 5900 K. To test the formulas derived in this work, we may for sake of illustration assume that all of the above parameters are exact. Because the parameters actually have uncertainties, the diffraction effects will also have a concomitant component of uncertainty, but we wish to consider only the component of uncertainty from using the asymptotic formulas derived here. The diffraction effects on total power reaching the radiometer are estimated to be a relative enhancement of 0.0012798062 (or about 0.13% excess flux). This same result was found to all digits shown by use of the small-A formulas (including terms up to and including those for $\tau = 10$) and by numerical evaluation of necessary integrals. For such accuracy, at present the integration required about 6000 Gauss–Chebyshev quadrature points for each integration over θ or θ', which demonstrates the practical advantage of the small-A formulas in the case of the double integration. Ten Gauss–Chebyshev quadrature points proved adequate for the integration over x in Eq. (6).

If one models diffraction effects on total power by using the effective-wavelength approximation, diffraction effects on spectral power at $\lambda_{\mathrm{eff}} = c_2\zeta(3)/[3\zeta(4)T] \approx 902.7920426$ nm are estimated to be a relative enhancement of 0.001165550 by the large-v formulas and by numerical evaluation of necessary integrals, with the added approximation of using 20 Gauss–Chebyshev quadrature points for the integration over x found in Eq. (6). (This quadrature is not fully converged, but it allows for a test of the asymptotic formulas.) Regarding the other aspects of integration, about 1000 quadrature points for each integration over θ or θ' were required to obtain all digits shown. These results also indicate the care that must be taken when one is considering use of the effective-wavelength approximation, and additional disadvantages such as poor convergence of integration by quadrature in Eq. (6). When used naïvely, as it would be in this case, the effective-wavelength approximation fails to realize near self-cancellation of oscillations in diffraction effects as a function of wavelength when one has a broadband source like the sun. On the other hand, including only the nonoscillatory term of the large-v formula for $L_B(v, w)$ proportional to v^{-1} gives 0.001271978, and including the nonoscillatory term proportional to v^{-3} in addition gives 0.001271946.

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